2 Linear dynamical systems. Topological conjugacies

♢ **2.1.** Recall that I defined in class *a one-parameter subgroup* of a matrix group G to be the image of a continuous group homomorphism $\gamma\colon \mathbf{R} \longrightarrow G$ (that is $\gamma(t+s)=\gamma(t)\gamma(s)$). The following theorem was stated:

Theorem 2.1. *If $\gamma\colon \mathbf{R} \longrightarrow GL(n,\mathbf{R})$ is a one-parameter subgroup then*

$$\gamma(t)=e^{At}.$$

I gave a proof assuming that γ is differentiable, but this is not necessary!

Give a full proof of this theorem assuming only continuity for $n=1$. (For a full proof of this theorem you will need a notion of a logarithm of a matrix.)

♢ **2.2.** Recall that a continuous dynamical system $\Phi\colon \mathbf{R}\times X \longrightarrow X$ satisfies the properties $\Phi(0,x)=x$ for all $x\in X$ and $\Phi(t+s,x)=\Phi(t,\Phi(s,x))$ for all $t,s\in\mathbf{R}, x\in X$. Two dynamical systems Φ and Ψ are called *topologically conjugate* if there exists a homeomorphism h such that $h(\Phi(t,x))=\Psi(t,h(x))$. Prove the following proposition.

Proposition 2.2. *Let $h\colon X\longrightarrow X$ be a topological conjugacy of two dynamical systems Φ,Ψ on X. Then*

(i) The point $\hat{x}$ is an equilibrium point of Φ if and only if $h(\hat{x})$ is an equilibrium point of Ψ.

(ii) the solution $\Phi(\cdot,x_0)$ is T-periodic if and only if $\Psi(\cdot,h(x_0))$ is T-periodic.

♢ **2.3.** Show that any smooth dynamical system $\Phi\colon \mathbf{R}\times\mathbf{R}^k \longrightarrow \mathbf{R}^k$ can be considered as the solution to ODE $\dot{x}=f(x)$, $x(t)\in\mathbf{R}^k$, where $f(x)=\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x)\big|_{t=0}$.

♢ **2.4.** Construct explicit topological conjugacy between a stable focus and a stable node on the plane.

♢ **2.5.** Consider scalar ODEs $\dot{x}=f(x)$, $\dot{x}=g(x)$, $x(t)\in\mathbf{R}$. Naturally they are called topologically equivalent if there is a homeomorphism h that takes the orbits of one to the orbits of another and preserves the sense of direction in time (note that this definition is somewhat more general than the one used in class). Prove

Theorem 2.3. *Two scalar autonomous ODE with a finite number of equilibria are topologically equivalent if and only if they have the same orbit structure (they have the same number of equilibria, with the same stability properties, in the same order).*

Math 861: *ODE II* by Artem Novozhilov©
e-mail: artem.novozhilov@ndsu.edu. Fall 2018